Logic and the Methodology of Science
November 2003 Preliminary Exam

August 23, 2005

1. Let $\mathcal{L}$ be a first-order language. We say that the $\mathcal{L}$-structure $\mathfrak{M}$ is *pseudofinite* if for every $\mathcal{L}$-sentence ϕ if $\mathfrak{M} \models \phi$, then there is a finite $\mathcal{L}$-structure $\mathfrak{N}$ with $\mathfrak{N} \models \phi$.

Show that if $\mathfrak{M}$ is pseudofinite and $f : M \to M$ is an $\mathcal{L}_M$-definable function from the universe of $\mathfrak{M}$ back to itself and f is surjective, then f is bijective. [NB: f may be defined with parameters from M.]

2.

(a) Show that there is a Δ^0_2 total function $f\colon \omega \to \omega$ such that whenever g is a partial recursive function on ω, then $\exists n \forall m \geq n (m \in \mathrm{dom}(g) \to g(m) \leq f(m))$.

(b) Show that if f is as in part (a), then every r.e. set is recursive in f.

3. Let $\mathcal{L}$ be the first order language having one n-ary relation symbol R_X for each natural number n and set $X \subseteq \omega^n$, and no other nonlogical symbols. Let $\mathfrak{A}$ be the $\mathcal{L}$ structure with universe ω such that $R_X^{\mathfrak{A}} = X$ for all X.

(a) Show that if $\mathfrak{B}$ is a reduct of $\mathfrak{A}$ to a countable sublanguage of $\mathcal{L}$ containing $R_<$, then the theory of $\mathfrak{B}$ is not ω-categorical. (Here $<$ is the usual order on ω.)

(b) Show that the theory of $\mathfrak{A}$ itself is ω-categorical.

4.

(a) Outline a proof that the theory of rings is not decidable.

(b) Show that the set $\mathcal{V}$ of all valid formulae in the language of rings is not recursive.

5. Let T be axiomatizable; that is, let T be an r.e. theory in a recursive language $\mathcal{L}$. In the language of Peano arithmetic (PA), let φ be a sentence which naturally expresses that T has a finite model. Suppose $\mathsf{PA} \cup \{\varphi\}$ is consistent. Show that T has a model.

6.

(a) Show that there is no r.e. set $A \subseteq \omega^2$ such that $\{A_n \mid n \in \omega\} = \{B \subseteq \omega \mid B \text{ is infinite and recursive }\}$. (Here $A_n = \{m \mid (n,m) \in A\}$.)

(b) Show that there is a r.e. set $A \subseteq \omega^2$ such that $\{A_n \mid n \in \omega\} = \{B \subseteq \omega \mid B \text{ is recursive }\}$.

7.

(a) Let $f\colon \omega \times \omega \to \omega$ be total and recursive. Show that there is a total recursive $g\colon \omega \to \omega$ such that

$$W_{g(n)} = W_{f(n,g(n))}$$

for all $n \in \omega$. (Here W_e is the e^{th} r.e. set in some standard enumeration.)

(b) Let $A = \{W_e \mid \{e\} = W_e\}$. Show that for all r.e. sets B, C there is a total recursive function f such that for all $n \in \omega$, $n \in (B \setminus C)$ iff $f(n) \in A$.

8. Let $\mathcal{L}$ be a countable language with no function or constant symbols. Recall that a sentence is *universal* if it is of the form $(\forall x_1)\cdots(\forall x_n)\theta(\vec{x})$ where θ is quantifier-free. Recall also that a sequence $\langle a_i \mid i \in \omega\rangle$ of elements of $M = |\mathfrak{M}|$ is *indiscernible* for $\mathfrak{M}$ if for any natural number n, any $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$, and pair of increasing n-tuples $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ of natural numbers, we have $\mathfrak{M} \models \psi[a_{i_1}, \ldots, a_{i_n}]$ iff $\mathfrak{M} \models \psi[a_{j_1}, \ldots, a_{j_n}]$.

(a) Show that if φ is a universal sentence of $\mathcal{L}$ and φ has an infinite model, then φ has a model $\mathfrak{M}$ such that $|\mathfrak{M}| = \{a_n \mid n < \omega\}$, for some sequence $\langle a_n \mid n < \omega\rangle$ which is indiscernible for $\mathfrak{M}$.

(b) Show that if $\mathcal{L}$ is recursive, then

$$\{\varphi \mid \varphi \text{ is a universal sentence which has an infinite model}\}$$

is recursive.

9. Let $\mathcal{L}$ be a countable language having a unary predicate symbol P, and possibly other nonlogical symbols. Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable, ω-homogeneous $\mathcal{L}$-structures which realize the same types. Suppose that $\mathfrak{A} \prec \mathfrak{B}$, that $\mathfrak{A} \neq \mathfrak{B}$, and that $P^{\mathfrak{A}} = P^{\mathfrak{B}}$. Show that there is a $\mathfrak{C}$ such that $\mathfrak{B} \prec \mathfrak{C}$ and $P^{\mathfrak{C}} = P^{\mathfrak{B}}$, and the universe of $\mathfrak{C}$ has cardinality ω_1.

[Recall that a structure $\mathfrak{A}$ is *ω-homogeneous* iff whenever $n < \omega$ and $a_1, ..., a_n, b_1, ..., b_n \in |\mathfrak{A}|$ and

$$(\mathfrak{A}, a_1, ..., a_n) \equiv (\mathfrak{A}, b_1, ..., b_n),$$

then there is an automorphism π of $\mathfrak{A}$ such that $\pi(a_i) = b_i$ for all $i \leq n$.]