Logic and the Methodology of Science
Autumn 2002 Preliminary Exam
Draft

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1. Let φ_e be the e^{th} partial recursive function in some standard enumeration, and let W_e be the domain of φ_e). Show that there is no partial recursive function ψ such that if W_e is recursive then $\psi(e)$ converges and $\varphi_{\psi(e)}$ is the characteristic function of W_e.

2. Let L be the language with $=$ and just one unary function symbol f. Let T be the L-theory whose axioms are: (1) $\forall x \forall y(f(x) = f(y) \rightarrow x = y$, (2) $\forall y \exists x(f(x) = y)$, and for each $n < \omega$, the axiom $(3)_n$:

$$\forall x(f^n(x) \neq x),$$

where $f^n(x) = f(f(...f(x)..))$, with f occuring n times. Show

(a) T is complete,

(b) T is decidable,

(c) T is not finitely axiomatizable.

3. Let Fin $:= \{e \in \omega : W_e \text{ is finite}\}$. Show that Fin is not recursively enumerable.

4. Let T be a theory in a countable language, and suppose T has infinite models. Show that there are models $\mathcal{A}_q$ of T, defined for q a rational number, so that

$$q < r \Rightarrow (\mathcal{A}_q \prec \mathcal{A}_r \wedge \mathcal{A}_q \neq \mathcal{A}_r).$$

5. Call a set $A \subseteq \omega$ *simple* if A is r.e., $\omega \setminus A$ is infinite, and for all infinite r.e. sets B, $A \cap B \neq \emptyset$.

(a) Let $f : \omega \rightarrow \omega$ be total, recursive, and injective. Show that

$$\text{ran}(f) \text{ is nonrecursive } \Leftrightarrow \{m \mid \exists n(m < n \wedge f(n) < f(m))\} \text{ is simple .}$$

(b) Let $A \subseteq \omega$, and suppose there is a partial recursive ψ such that whenever $W_e \cap A = \emptyset$, then $\psi(e)$ converges and $\psi(e) \notin A \cup W_e$. Show that A is not simple.

6. Call a structure $\mathcal{A}$ *homogeneous* iff whenever $n < \omega$ and $a_1, ..., a_n, b_1, ..., b_n \in |\mathcal{A}|$ and

$$(\mathcal{A}, a_1, ..., a_n) \equiv (\mathcal{A}, b_1, ..., b_n),$$

then there is an automorphism π of $\mathcal{A}$ such that $\pi(a_i) = b_i$ for all $i \leq n$.

(a) Show that if $\mathcal{A}$ and $\mathcal{B}$ are countable, homogeneous structures for the same language, and $\mathcal{A}$ and $\mathcal{B}$ realize the same n-types, for all n, then $\mathcal{A} \cong \mathcal{B}$.

(b) Let $\mathcal{A}$ be a countable structure for a countable language. Show there is a countable, homogeneous structure $\mathcal{B}$ such that $\mathcal{A} \prec \mathcal{B}$.

7. Let L be the language with one binary relation symbol.

(a) Show that if T is a consistent, recursively axiomatizable L-theory, then T has a model of the form (ω, R), where R is Δ^0_2.

(b) Show that if T is a consistent, decidable L-theory, then T has a model of the form (ω, R), where R is recursive.

(c) Give an example of a consistent, recursively axiomatizable L-theory with no model of the form (ω, R), for R recursive. [Hint: You may assume there is an infinite, recursive binary tree with no recursive infinite branch. Also, you are welcome to change L to the language of Peano Arithmetic, and prove the corresponding result there, if it helps you.]

8. Let Prov(x) be a standard formula in the language of Peano Arithmetic (PA) asserting that x is the Godel number of a sentence which is provable in PA. Let φ be a sentence in the language of PA, let $\check{\varphi}$ be the numeral for the Godel number of φ, and suppose

$$\mathsf{PA} \vdash \mathrm{Prov}(\check{\varphi}) \rightarrow \varphi.$$

Show that PA proves φ. [Hint: use Godel's theorem to show that $\mathsf{PA} \cup \{\neg\varphi\}$ is inconsistent.]