LOGIC AND THE METHODOLOGY OF SCIENCE PRELIMINARY EXAMINATION

1. Prove or disprove: For any uncountable well-ordered set $(X, <)$ there is a countable *well-ordered* set $(Y, <)$ for which $(X, <) \equiv (Y, <)$.

2. Suppose that $\mathcal{L}$ is a first-order language having only finitely many nonlogical symbols and that T is a theory in $\mathcal{L}$ having no uncountable models. Show that up to isomorphism T has only finitely many models.

3. Show that there is some $e \in \omega$ for which $(\forall x \in \omega) x \in W_e \leftrightarrow (x + e + 1) \in W_e$. Show, moreover, that any such W_e is recursive.

4. Let $\mathfrak{N}$ be a nonstandard model of Peano arithmetic. Show that there is an element $a \in \mathfrak{N}$ such that for any standard prime number p, p^p divides a and a/p^p is coprime to p.

5. Show that there is a total recursive function $f : \omega \to \omega$ such that for all $e \in \omega$ the set W_e is finite if and only if $\omega \setminus W_{f(e)}$ is finite.

6. Consider the structure (ω, S) where $S : \omega \to \omega$ is the successor function $x \mapsto x + 1$. Let $T := \mathrm{Th}(\omega, S)$ be the complete theory of this structure. How many 3-types (over $\varnothing$) are there relative to T? Describe all of the 3-types giving isolating formulas where possible.

7. Let $\mathrm{Pr}_{\mathrm{PA}}(x)$ be the usual formula which naturally expresses that the sentence encoded by x is provable from Peano arithmetic. Let $\phi(x)$ be a formula in the language of arithmetic in the one free variable x. Let Sub_ϕ be the definable (relative to Peano arithmetic) function which takes a number a and returns the code for the sentence obtained by substituting a for x in ϕ. Show that if $\mathrm{PA} \vdash (\forall z)(\mathrm{Pr}_{\mathrm{PA}}(\mathrm{Sub}_\phi(z)) \to \phi(z))$, then $\mathrm{PA} \vdash (\forall z)\phi(z)$.

8. Let $\mathcal{L}$ be a first order language and $\mathfrak{A}$ and $\mathcal{L}$-structure with universe A. Let $\mathcal{L}'$ be obtained from $\mathcal{L}$ by adjoining one new one place relation symbol $\mathbb{S}$. We say that $S \subseteq A$ is *implicitly definable* if there is an $\mathcal{L}'$ sentence σ for which $(\mathfrak{A}, S') \models \sigma \Leftrightarrow S = S'$.

Is it the case that whenever a set S is implicitly definable in some $\mathcal{L}$-structure, then it must be explicitly (*ie* in $\mathcal{L}$) definable? Prove that your answer is correct.

9. Show that there is a nonstandard model $\mathfrak{N}$ of Peano arithmetic having no proper elementary submodels.

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