GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE
PRELIMINARY EXAMINATION

There are eight questions. Partial credit may be assigned for substantially correct partially worked solutions. To pass, you need a score of roughly fifty percent, though the ultimate decision about the passing mark will be decided by the committee of graders.

1.

(a) Show there are disjoint Σ^0_2 sets A and B of natural numbers such that there is no Δ^0_2 set C with $A \subseteq C$ and $C \cap B = \emptyset$.

(b) Call a theory T in the language of PA Σ^0_1-*sound* just in case T proves no false existential sentences, i.e. whenever $T \vdash \exists v\theta$ where all quantifiers in θ are bounded, then $\exists v\theta$ is true in the standard model of PA. Show that there is no complete, Σ^0_1-sound T extending PA such that T is Δ^0_2.

2. Let $\leq_T$ be Turing reducibility, and $K := \{e \in \omega \mid e \in W_e\}$. Show that there is an A such that $\emptyset <_T A <_T K$.

3. Let T be the theory of $(\mathbb{Z}, <)$.

(a) Show that T is finitely axiomatizable.
(b) Show that T has a prime model.
(c) Show that T has a countable saturated model.

4. Let $f \colon N \to N$ be total recursive. Show that there is a formula $\varphi(v)$ in the language of PA such that

(a) $\mathsf{PA} \vdash \varphi(\bar{n})$ for each n, where $\bar{n}$ is the numeral for n, and
(b) letting $g(n)$ be the least Gödel number of a proof of $\varphi(\bar{n})$ in PA, we have $f(n) < g(n)$ for all n.

5. An element a of some model M is *definable* if there is a formula $\phi(x)$ for which $M \models (\forall x)(\phi(x) \leftrightarrow x = a)$.

(a) Show that $a \in M$ is definable if and only if for every elementary extension $N \succeq M$ and every automorphism $\sigma : N \to N$ one has $\sigma(a) = a$.
(b) Show by example (with proof) that it may happen that a is not definable but for every automorphism $\sigma : M \to M$ one has $\sigma(a) = a$.

6. Let $E := \{e \in \omega \mid W_e = \{x \in \omega : (\exists y \in \omega)\ y + y = x\}\}$. Compute the position of E within the arithmetic hierarchy.

7. Show that for any complete theory T having infinite models there is a model $M \models T$ and a descending chain of elementary submodels models $M_\alpha \models T$ (for $\alpha \in \omega + 1$) so that $M = M_0 \succneqq M_1 \succneqq M_2 \succneqq \cdots \succneqq \bigcap_{n=0}^{\infty} M_n = M_\omega$ and $M \cong M_\alpha$ for all $\alpha \in \omega + 1$.

8. Let L be a language having at least one constant symbol. Let T be a consistent L-theory which is axiomatized by universal sentences. Prove that the following two conditions on T are equivalent.

AP: T has the amalgamation property: For any three models $A \models T$, $B \models T$ and $C \models T$ with $A \subseteq B$ and $A \subseteq C$, there is a model $D \models T$ and embeddings $f : B \to D$ and $g : C \to D$ so that $f \upharpoonright A = g \upharpoonright A$.

QfI: T satisfies quantifier-free interpolation in the sense that for any three tuples of (disjoint) variables x, y, and z and quantifier-free formulae $\phi(x, y)$ and $\psi(y, z)$ if $T \vdash \phi(x, y) \to \psi(y, z)$, then there is a quantifier-free formula $\vartheta(y)$ so that $T \vdash \phi(x, y) \to \vartheta(y)$ and $T \vdash \vartheta(y) \to \psi(y, z)$.

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