GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE
PRELIMINARY EXAMINATION

There are nine questions. Partial credit may be assigned for substantially correct partially worked solutions. To pass, you need a score of roughly fifty percent, though the ultimate decision about the passing mark will be decided by the committee of graders.

1. Let A be an infinite r.e. set, and let R be an r.e. partial order of A. Suppose R is directed, that is, for all $a, b \in A$, there is c such that $R(a, c)$ & $R(b, c)$. **Show** that there is a total recursive function f such that $(\forall a \in A)(\exists n)R(a, f(n))$.

2. **Show** that for any infinite model M and cardinal κ there is some elementary extension $N \succeq M$ whose automorphism group has size at least κ. [Hint: Use Ehrenfeucht-Mostowski models.]

3. Let W_e be the e-th r.e. set in a standard enumeration. Put

$$R_e = \{\langle a, b\rangle \in \omega^2 \mid 2^a 3^b \in W_e\} .$$

Show that the set

$$\mathsf{L} := \{e \in \omega \mid R_e \text{ is a linear order of } \omega\}$$

is Π^0_2-complete.

4. Let $\mathcal{L}$ be a countable first-order language, T an $\mathcal{L}$-theory and $\mathcal{L}' := \mathcal{L}(\{P_i \ : \ i \in \omega\})$ an expansion of $\mathcal{L}$ by countably many new predicate symbols. Suppose that $\Sigma(x)$ is a set of $\mathcal{L}$ formulae and that for each $n \in \omega$ there is an $\mathcal{L}(\{P_i \ : \ i < n\})$-structure $\mathfrak{M}_n$ for which $\mathfrak{M}_n \models T$ but $\mathfrak{M}_n$ omits $\Sigma(x)$. **Show** that there is an $\mathcal{L}$-structure $\mathfrak{M} \models T$ which omits $\Sigma(x)$.

5. **Show** that there is no partial recursive function ψ such that whenever W_e is finite, then $\psi(e)$ is defined, and $|W_e| \leq \psi(e)$.

6. Let $\mathcal{L} = \mathcal{L}(<)$ be the theory whose signature consist of a single binary relation symbol $<$. By "the theory of finite linearly ordered sets" we mean the theory

$$\mathsf{T}_{flo} := \{\psi \in \mathcal{L} \mid (X, <) \models \psi \text{ for every (nonempty) finite linear order } (X, <)\}$$

Show that T_{flo} is decidable.

7. Let $\mathfrak{N} \models \mathrm{PA}$ be a model of Peano Arithmetic and $a \in |\mathfrak{N}|$. Let $I = \{b \mid \mathfrak{N} \models b \leq a\}$, and let $\leq^*$ be a linear order of I such that $\leq^*$ is definable from parameters over $\mathfrak{N}$. **Show** that $(I, \leq^*)$ is *not* isomorphic to $(\omega, \leq)$.

8. Let T be a consistent, recursively axiomatizable theory. **Show** that T has a model $\mathfrak{A}$ such that $\mathrm{Th}(\mathfrak{A})$ is Δ^0_2.

9). For T a recursively axiomatizable theory, let Con_T be a natural formalization of "T is consistent".

(a) Let T be a recursively axiomatizable theory in the language of PA, with $\mathrm{PA} \subseteq T$. Suppose that $T \vdash \varphi$, where φ is a Π_1 sentence. **Show** that if T is consistent, then φ is true.

(b) Let φ be a Π_1 sentence such that $\mathrm{PA} + \neg\mathrm{Con}_{\mathrm{PA}} \vdash \varphi$. **Show** that $\mathrm{PA} \vdash \varphi$. [Hint: You may assume that the proof of the second incompleteness theorem for PA can be formalized in PA. Use this to show $\mathrm{PA} + \mathrm{Con}_{\mathrm{PA}} \vdash \varphi$.]

Date: 8 June 2012.