GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE PRELIMINARY EXAMINATION

1. Suppose that $\psi(y)$ and $\phi(x,y)$ are two formulae in the language of arithmetic, $\mathcal{L}(+,\times,\leq,0,1)$, and that the partial type

$$\Sigma(x) := \mathrm{Th}(\mathbb{N}) \cup \{\phi(x,n) : \mathbb{N} \models \psi(n)\} \cup \{\neg\phi(x,n) : \mathbb{N} \models \neg\psi(y)\}$$

is consistent. **Prove** that if ${}^*N \supsetneq \mathbb{N}$ is a proper extension of the natural numbers and ${}^*N \equiv \mathbb{N}$, then there is a $b \in {}^*N$ for which ${}^*N \models \Sigma(b)$.

2. Show that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every arithmetically definable set can be computed from any function g for which $(\forall n)[g(n) \geq f(n)]$.

3. Prove that $\mathrm{Th}(\mathbb{Z},<)$ is decidable.

4. By a finitely branching rooted tree, we shall mean a set T given together with a partial ordering $\leq$ for which

- there is an element $t_0 \in T$ such that $(\forall t \in T)[t_0 \leq t]$,
- for each $t \in T$ the initial set $\{s \in T : s \leq t\}$ is a finite set linearly ordered by the restriction of $\leq$, and
- for each $t \in T$ the set of immediate successors of t is finite.

By a *recursive* finitely branching rooted tree we shall me such a tree $(T,\leq)$ for which T is the set $\mathbb{N}$ of natural numbers and $\leq$ is recursive subset of $\mathbb{N}^2$.

Prove that **(1)** if $(T,\leq)$ is an infinite, finitely branching rooted tree, then there is an infinite subset $S \subseteq T$ which is linearly ordered by $\leq$, but **(2)** there is a recursive finitely branching rooted tree for which there is no infinite recursive set $S \subseteq T$ linearly ordered by $\leq$.

5. We say that the theory T eliminates the quantifier $\exists^\infty$ if for each formula $\phi(x;y) = \phi(x_1,\ldots,x_n;y_1,\ldots,y_m)$ there is a formula $\vartheta(y)$ so that for any model $M \models T$ we have $M \models \vartheta(b) \iff \{a \in M^n : M \models \phi(a;b)\}$ is infinite . **Show:**

- T eliminates $\exists^\infty$ if and only if for each formula $\phi(x;y) = \phi(x_1,\ldots,x_n;y_1,\ldots,y_m)$ there is a number $N = N(\phi)$ so that for any model $M \models T$ and parameter $b \in M^m$ the set $\{a \in M^n : M \models \phi(a;b)\}$ is finite, it has size at most N.
- If T is countable and T does *not* eliminate $\exists^\infty$, then there is an uncountable model $M \models T$ and a countably infinite definable (with parameters) set $X \subseteq M$.

6. Prove that if $X \subseteq \mathbb{N}$ is an infinite recursively enumerable set, then there is a recursive function $f : X \to X$ so that f has no fixed points but $f \circ f = \mathrm{id}_X$.

7. Recall that a model M is *universal* if for every $N \equiv M$ with $|N| \leq |M|$ there is an elementary embedding $g : N \overset{\preceq}{\hookrightarrow} M$. **Give** an example (with a proof that it is an example) of a universal model which is *not* saturated.

8. Does there exist a consistent, recursive extension T of PA for which $T \vdash \neg\,\mathrm{Con}(T)$? Justify your answer.

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