Logic and the Methodology of Science
February 2005 Preliminary Exam

August 23, 2005

1. Let $\varphi(v)$ be a formula in the laguage of Peano Arithmetic (PA).

(a) Suppose that $\varphi(v)$ is Σ_1, and $\mathsf{PA} \vdash \exists v \varphi(v)$. Show that $\mathsf{PA} \vdash \varphi(\bar{n})$ for some numeral $\bar{n}$.

(b) Give an example of a formula $\varphi(v)$ such that $\mathsf{PA} \vdash \exists v \varphi(v)$, but for all n, PA does not prove $\varphi(\bar{n})$.

(c) Suppose $\varphi(v)$ is Σ_1 and T is a consistent extension of PA such that $T \vdash \exists v \varphi(v)$. Does it follow that $T \vdash \varphi(\bar{n})$ for some n.

2. Show there is a one-one 2-ary partial recursive function Ψ such that for every one-one 1-ary partial recursive f, there is an e such that for all i $f(i) = \Psi(e, i)$.

3. Let L be a finite language, and let T be an axiomatizable L-theory. Fix a recursive enumeration of T, and let T_n be the first n sentences of T in this enumeration. Suppose $M \models \mathsf{PA}$ is such that

$$M \models \mathrm{Con}(T_n)$$

for all n. (On the right hand side, "T_n" should be interpreted as the numeral for the Godel number of T_n.) Show that M interprets a model of T; that is, there is a model of T whose universe, functions, and relations are all definable from parameters over M.

4. Let E be an r.e. equivalence relation on ω, and suppose E is not recursive. Show

(a) E has infinitely many equivalence classes,

(b) for each n, there are infinitely mant equivalence classes whose cardinality is different from n.

5. Let A be an r.e. set, and $B = \{e | W_e = A\}$. Show that either B is a Δ^0_2 set, or B is a complete Π^0_2 set.

6.

(a) A *graph* is a set with an irreflexive, symmetric binary relation. Show there is a graph $G = (V, E)$ such that whenever J and K are disjoint finite subsets of V, then there is an $a \in G$ such that

$$\forall b \in J(aEb) \text{ and } \forall b \in K(\neg aEb).$$

(b) Show that if G is a graph as in part (a), then the theory of G is decidable.

7. Show that the theory of $(\mathbb{Q}, +)$ is decidable.

8. Let T be the theory of $(\mathbb{Z}, +)$. How many countable models (up to isomorphism) does T have?

9. Let T be a complete theory in a countable language. Show that the following are equivalent:

(a) T has a prime model $\mathcal{A}$ such that there is a $\mathcal{B} \prec \mathcal{A}$ with $\mathcal{B} \neq \mathcal{A}$,

(b) T has an uncountable atomic model.